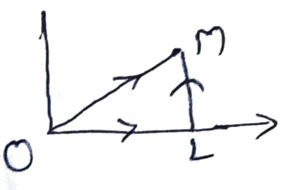
* Integrate z^2 along OM and also along path OLM consisting of 2 straight line segments OL and OM where O is the origin, L is the point $z=3$, M is the point $z=3+i$. Hence show that integral of z^2 along the closed path OLMO is zero.



Soln

Along OM

$x = 3y \Rightarrow dx = 3dy$

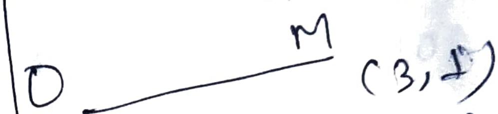
$$\therefore \int_{OM} z^2 dz = \int_{y=0}^{1} (x+iy)^2 (dx + i\,dy)$$

$$= \int_0^1 (3y+iy)^2 (3dy + i\,dy)$$

$$= (3+i)^3 \int_{y=0}^{1} y^2 dy = \frac{(3+i)^3}{3}$$

at M, $z = 3+i = x+iy$

$\Rightarrow x = 3,\ y = 1$



eqn of OM is

$$\frac{x-0}{0-3} = \frac{y-0}{0-1}$$

$$\Rightarrow \frac{x}{3} = \frac{y}{1} \Rightarrow x = 3y$$

$\therefore$ along OM, y varies from 0 to 1

(alternatively x varies from 0 to 3)

$$= \frac{1}{3}\left[27 - i + 27i - 9\right] = \frac{1}{3}\left[18 + 26i\right] = 6 + \frac{26}{3}i$$

Along OLM

$$\int_{OLM} z^2 dz = \int_{OL} z^2 dz + \int_{LM} z^2 dz$$

$$= \int_{x=0}^{3} x^2 dx + \int_{y=0}^{1} (3+iy)^2 \cdot i\,dy$$

$$= 9 + i\left[\frac{(3+iy)^3}{3i}\right]_0^1 = 9 + \frac{(3+i)^3}{3} \ldots = 6 + \frac{26}{3}i$$

at L ($z=3$)

$\Rightarrow x = 3, y = 0$

$\therefore$ along OL,

$x = 0$ to 3

and $y = 0$

$\Rightarrow dy = 0$

along LM,

$y = 0$ to 1, $x = 3$

$\Rightarrow dx = 0$

$$= 9 + i \int_{y=0}^{1} (3+iy)^2 \, dy = 9 + i \int_{0}^{1} (9 - y^2 + 6iy) \, dy$$

$$= 9 + i \left[\frac{(3+iy)^3}{3i} \right]_0^1 \qquad = 9 + i \left[9 - \frac{1}{3} + 3i \right]$$

$$= 9 + 9i - \frac{i}{3} - 3$$

$$= 6 + \frac{26i}{3}$$

$$\cancel{= 9 + \frac{1}{3}(3+i)^3 = 9 + \frac{1}{3}(18+26i)}$$

Now $\displaystyle\int_{OLMO} = \int_{OLM} + \int_{MO} = \int_{OLM} - \int_{OM}$

$$= 6 + \frac{26}{3}i - \left(6 + \frac{26}{3}i\right) = \underline{\underline{0}}$$